New Simplified Models and Formulations for Turbulent Flow and Convection

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Expression of the time-averaged equations for the conservation of momentum and energy in terms of the fractions of the local shear stress and heat flux density, respectively, due to turbulence is shown to result in greatly simplified representations for fully developed flow and convection in round tubes and parallel-plate channels as compared to those in terms of the eddy viscosity, eddy conductivity, or mixing length. On the other hand, the turbulent Prandtl number is shown to be a fundamental characteristic of flow and convection rather than simply an artifact of the eddy diffusional model. The new simplified representations provide the basis for improved predictions of the velocity distribution, the friction factor, the temperature distribution, and the Nusselt number. The adaption of these new representations for other geometries and for developing convection is discussed briefly.

Introduction

This article is primarily concerned with models, descriptions, and formal solutions for flow and forced convection in the fully turbulent regime in a round tube and between identical parallel plates. New, simplified formulations are proposed for the more accurate and expeditious predictions of the velocity distribution, the temperature distribution, the friction factor, and the Nusselt number. These formulations are based directly on the time-averaged equations for the conservation of momentum and energy and do not depend on any mechanistic postulates or empiricism. Their implementation for numerical calculations does necessarily invoke some empiricism, but generally less than that required by prior models and formulations.

Time averaging the equations of conservation generates unknown quantities, such as $\overline{u'v'}$ and $\overline{T'v'}$. The history of analysis in applied turbulent flow and convection consists primarily of the postulation and utilization of differential models for these two unknown quantities, the most widely used being the eddy-diffusivity model of Boussinesq (1877) and the mixing-length model of Prandtl (1925). Although these two models have been applied with reasonable success to make predictions for turbulent flow and convection, they are viewed with suspicion or distaste by most analysts for three primary reasons. First, the mechanistic concepts upon which they are based are demonstratably false; second the parameters they introduce appear to have no physical basis; and third the evaluation of these artificial parameters poses a demand for experimental data of greater accuracy and wider scope than are available even today.

Patankar and Spalding (1967) proposed, as a means of alleviation of the second and the third of these shortcomings, the prediction of the eddy viscosity or the mixing length from equations of conservation for the kinetic energy of turbulence κ and the rate of dissipation of the energy of turbulence ϵ. This κ-ϵ model incorporates a significant degree of empiricism and approximation, but was expected to have the compensatory merit of great generality. Unfortunately the predictions of the κ-ϵ model and its many variants and modifications have not proven to be as accurate or geometry-independent as was originally anticipated.

Churchill and Chan (1995b) and Churchill (1996) cited a fourth shortcoming of the eddy-viscosity and mixing-length models for flow, namely their fundamental failure for many conditions. The mixing length is shown to be unbounded at the center line of a round tube and at the central plane between parallel plates, while both the mixing length and the eddy viscosity are found to be unbounded at one location and negative over an adjacent region in any channel in which the shear stress is unequal on opposing walls. Such anomalous behavior of the eddy viscosity and the mixing length occurs, for example, in all annuli, in flow between parallel plates of unequal roughness, and in combined forced and wall-induced flows. The models are thus, strictly speaking, inapplicable in all such asymmetrical flows. Moreover, since the κ-ϵ model necessarily functions by predicting the eddy viscosity or the mixing length, it is also inapplicable for these same geometries and conditions. Because the eddy-viscosity, the mixing-length, and the κ-ϵ models usually produce reasonable approximations for integral behavior (for example, for the friction factor and the Nusselt number) in spite of the local singularities, this particular shortcoming has seldom been mentioned or even recognized.

The κ-ϵ-$\overline{u'v'}$ model, which functions by predicting the turbulent shear stress and the local velocity directly without reference to an eddy viscosity or a mixing length, remains applicable for these asymmetric flows (for example, Hanjalić and Launder, 1972a), but invokes considerable empiricism and approximation, and yields accurate predictions only for the turbulent core.

It was generally presumed in the past that turbulent shear flows could not be simulated numerically because of the wide range in scale and the chaotic nature of the turbulent fluctuations in velocity. The presumably impossible task has, however, been accomplished in the recent decade by clever modeling, the recognition that the shear stress produced by the smallest fluctuations is negligible relative to the viscous stress, and the availability of computers of ever-increasing capability. The results of such *direct numerical simulations* (DNS), which appear to be convergent and in good agreement with experimental measurements (for example, Rutledge and Sleicher, 1993; Lyons et al., 1991), are yet limited to two fully developed flows, namely, between parallel plates and along a flat plate, and in both instances to the very lowest range of fully turbulent motion. Also, the results of the DNS are in the form of discrete numerical values and thereby provide no direct functional insight. Because of these limitations and because of the truly great computational requirements, this methodology is not yet and may never be of direct utility for design or control. The results of these direct simulations are nevertheless an invaluable resource for the construction and critical testing of approximate models and correlating equations of more general scope.

The great power of dimensional and speculative analysis has long been recognized, but this technique has seldom been fully exploited for modeling turbulent flow and convection. The combination of numerical values from the direct simulations together with functional forms from extended asymptotic and speculative analyses provides the basis for a significant improvement in models and correlating equations for the turbulent regime. Such models and correlating equations have recently been developed by Churchill and Chan (1994, 1995a,b) for flow in a round tube and between identical parallel plates. The initial objective of the current work was to extend these models or to develop new models of this type for energy and mass transfer. An unexpected development was the discovery of an improved model for momentum transfer as well. The new model for momentum transfer, just as the earlier ones of Churchill and Chan, is free of any heuristic variables such as the eddy viscosity. It was presumed that the corresponding heuristic variables for heat transfer, such as the eddy conductivity and the turbulent Prandtl number, could also be avoided in the new formulations. The result proved to be a surprise.

In the interests of simplicity, invariant physical and thermal properties are postulated throughout. A secondary flow, normal to the primary direction of flow, occurs in the turbulent regime in all two-dimensional channels, including open ones. To avoid that complication, consideration herein is limited to confined flow in one-dimensional channels and unconfined flow along a semi-infinite flat plate.

Momentum transfer will be considered separately and first, since the results serve as the starting point in all instances for the development of models and solutions for convective thermal transfer. Flow in a smooth round tube is examined first, then convection in a uniformly heated tube, and finally the possible extension of those results to other geometries and conditions.

Momentum Transfer in a Round Tube

Earliest models

The simplest model for fully developed turbulent flow in a smooth round tube is that of uniformity of the time-averaged flow over the cross section. This model implies independence of the friction factor from the Reynolds number, which is actually not a bad approximation for some conditions and applications.

The next simplest model, which may be derived by means of elementary dimensional analysis, is

$$f = \phi\{Re\}. \tag{1}$$

The power-series method of deriving Eq. 1 has often been misinterpreted to imply that

$$f = ARe^{\alpha} \tag{2}$$

Blasius (1913) appeared to give support for such a model by fitting experimental data for a limited range of values of Re reasonably well with the expression

$$f = 0.0791/Re^{1/4}. \tag{3}$$

C. Freeman suggests that had Blasius had access to the more extensive experimental data of J. Freeman (1941) that were obtained in 1892 but not published in the open literature until half a century later, he would probably have developed some other, better correlating equation and thereby changed the history of fluid mechanics. It can be inferred from Eq. 3 that the corresponding velocity distribution is

$$\frac{u}{u_m} = \frac{60}{49}\left(\frac{y}{a}\right)^{1/7}. \tag{4}$$

Equation 4 fails to yield a finite gradient at the wall and thereby a finite shear stress, and it also fails to yield a gradient of zero at the center line, implying related shortcomings of Eq. 3 in both instances. Equations 3 and 4 have nevertheless been widely used for predictions of flow, reaction, heat transfer, and mass transfer in round tubes.

Basic relationships and representations

Most of the mechanistic models for turbulent flow are based on the time-averaged form of the partial differential equations for the conservation of mass and momentum. The time-averaged expression for momentum transfer in the radial direction in fully developed flow in a round pipe may be combined with an overall force balance to obtain

$$\tau_w\left(1-\frac{y}{a}\right)=\mu\frac{du}{dy}-\rho\overline{u'v'}. \tag{5}$$

Barenblatt and Goldenfeld (1995) have recently questioned the existence of a fully developed state in a round tube, but for all practical purposes a close approach to one appears to be attained in a relatively short distance from the entrance. The existence of a fully developed state will be implied in all that follows.

The price of the great simplification attained by time-averaging is the appearance of unknown quantities such as $\rho\overline{u'v'}$ in Eq. 5. Most of the generally accepted mechanistic models for turbulent flow involve the postulate of an arbitrary differential expression for $\rho\overline{u'v'}$. Before examining such models the behavior of $\rho\overline{u'v'}$ will be described. For this purpose Eq. 5 has generally been reexpressed in the following dimensionless form:

$$1-\frac{y^+}{a^+}=\frac{du^+}{dy^+}+\left(\overline{u'v'}\right)^+, \tag{6}$$

where the quantity $(\overline{u'v'})^+\equiv-\rho\overline{u'v'}/\tau_w$ is the ratio of the local turbulent shear stress to the total shear stress at the wall. Churchill and Chan (1995b) advocated and Churchill and Chan (1994, 1995a) utilized this form to develop new correlations. However, the modified form

$$\left(1-\frac{y^+}{a^+}\right)\left[1-\left(\overline{u'v'}\right)^{++}\right]=\frac{du^+}{dy^+} \tag{7}$$

where

$$\left(\overline{u'v'}\right)^{++}\equiv-\frac{\rho\overline{u'v'}}{\tau}=\left(\overline{u'v'}\right)^+\Big/\left(1-\frac{y^+}{a^+}\right) \tag{8}$$

is the fraction of the shear stress at any location due to turbulence, appears to be somewhat more convenient in that the known variation of the total shear stress with y^+/a^+ is factored out of the dependent variable $(\overline{u'v'})^+$.

Since u^+ is known to be a function of y^+ and a^+, it follows from Eq. 7 that $(\overline{u'v'})^{++}$ is a function of the same two independent dimensionless variables. The quantities u, $\overline{u'v'}$, and τ_w are directly measurable, but the precision and accuracy of the resulting values in the literature for du^+/dy^+ and $(\overline{u'v'})^{++}$ or the equivalent are unsatisfactory to this day. That is why the prediction of precise and presumably accurate values by direct numerical simulations, even for a limited range of conditions, is of such great importance. The local shear stress τ is not easily measured, but, as implied by Eq. 8, its value follows exactly from the value at the wall by virtue of

$$\tau=\tau_w\left(1-\frac{y^+}{a^+}\right). \tag{9}$$

Churchill and Chan (1995a) devised a very accurate and general correlating equation for $(\overline{u'v'})^+$ based on asymptotic solutions, experimental data, and the aforementioned DNS values. Their expression may be rephrased in terms of $(\overline{u'v'})^{++}$ as follows:

$$\left(\overline{u'v'}\right)^{++}=\left(\left[\frac{0.7\left(\frac{y^+}{10}\right)}{1-\frac{y^+}{a^+}}\right]^n+\left|\exp\left\{\frac{-2.5}{y^+}\right\}-\frac{2.5}{a^+}\left(1+\frac{4y^+}{a^+}\right)\right|^n\right)^{1/n}, \tag{10}$$

where $n=-8/7$ for $a^+>500$ and decreases in magnitude to $n=-1$ as $a^+\to 150$. The characteristic quantity $1-(\overline{u'v'})^{++}$ of Eq. 7, as calculated from Eq. 10 is seen in Figure 1 to fall rapidly and monotonically from unity as y^+ exceeds 30 and to approach $15/a^+$ as $y^+\to a^+$, whereas $(\overline{u'v'})^+$, as shown in Figure 3 of Chan and Churchill (1995a), arises from zero at the wall to a maximum value less than unity and then decreases to zero at the centerline.

Equation 7 may be integrated formally to obtain

$$u^+=\frac{a^+}{2}\int_{R^2}^{1}\left[1-\left(\overline{u'v'}\right)^{++}\right]dR^2. \tag{11}$$

Here, for simplicity $R=1-(y^+/a^+)$ has been introduced as a variable. Numerical integration of Eq. 11 using values of $(\overline{u'v'})^{++}$ from Eq. 10 is presumed to produce more accurate

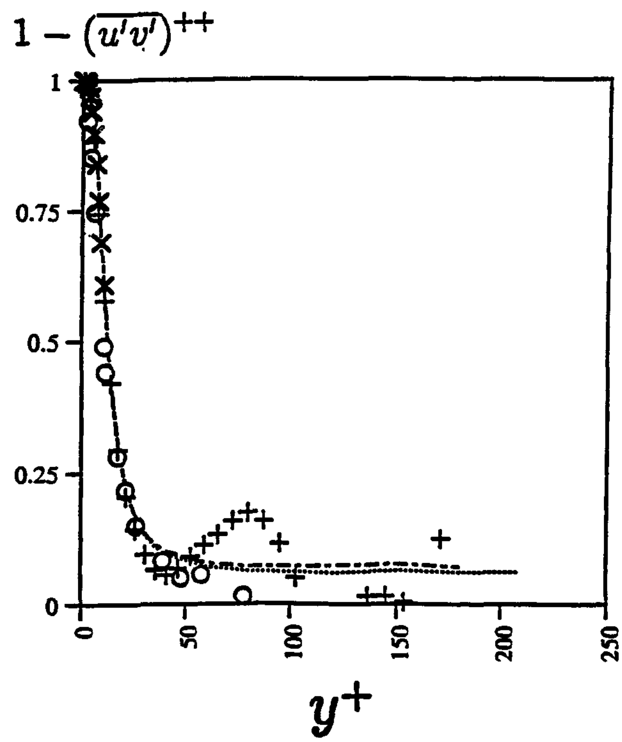

Figure 1. Fraction of the local shear stress due to viscosity.

Simulated values, $b^+=180$ (× Kim et al. (1987); + Rutledge and Sleicher (1993)); experimental values, $b^+=208$ (○ Eckelmann (1974)); predictions, Eq. 10 with $n=-1$ (— · —, $b^+=180$; · · · · · ·, $b^+=208$).

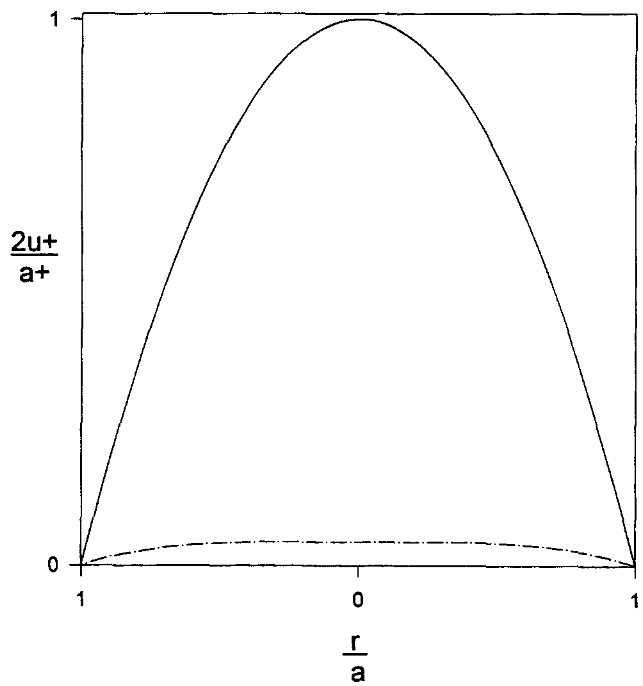

Figure 2. Velocity distributions in a round tube at $a^+ = 180$, according to Eq. 12.

(——— laminar flow; - - - - turbulent flow).

values of $u^+\{y^+, a^+\}$ than the direct use of the corresponding correlating equation of Churchill and Chan (1995a) for u^+, since Eq. 10 involves one less empirical coefficient, and since integration is a smoothing process that usually reduces the impact of any error in the integrand. Such an improvement is of course at the expense of carrying out a numerical integration in each instance.

Equation 11 may be integrated analytically in part to obtain the following:

$$u^+ = \frac{a^+}{2}(1-R^2) - \frac{a^+}{2}\int_{R^2}^{1} \overline{(u'v')}^{++}\, dR^2, \qquad (12)$$

which reveals that the velocity distribution is reduced at each radius from that for fully developed laminar flow at the same value of a^+ by virtue of the remaining integral term. This behavior is illustrated in Figure 2, in which the region between the two curves represents the contribution of the turbulence to the velocity distribution. Figure 2 differs from the conventional comparisons of the laminar and turbulent velocity distributions in that the velocity distributions are plotted for the same value of a^+, that is, for the same value of the pressure gradient, rather than for the same value of the Reynolds number, that is, for the same rate of flow, as for example, in figure 7-1 of Knudsen and Katz (1953).

An expression for the mixed-mean velocity, and hence for the Fanning friction factor $f = 2/(u_m^+)^2$ may be obtained by utilizing the expression for u^+ in terms of $\overline{(u'v')}^{++}$ and integrating by parts. This result may be expressed as

$$u_m^+ = \frac{a^+}{4}\int_0^1 \left[1 - \overline{(u'v')}^{++}\right] dR^4 \qquad (13)$$

or

$$u_m^+ = \frac{a^+}{4}\left[1 - \int_0^1 \overline{(u'v')}^{++}\, dR^4\right]. \qquad (14)$$

The integration with respect to R^4 in both Eqs. 13 and 14 indicates that the values of the integrand very near the wall are most influential in determining u_m^+. Equation 13, together with Eq. 10, is perhaps the most accurate source of numerical values of $u_m^+\{a^+\}$ and hence of $f\{a^+\}$ in the literature. Churchill and Chan (1994) used the equivalent of this formulation to construct the following very accurate and very general correlating equation for the friction factor

$$\left(\frac{2}{f}\right)^{1/2} = u_m^+ = 1.989 - \frac{161.2}{a^+} + \left(\frac{47.6}{a^+}\right)^2 + 2.5\ln\{a^+\}. \quad (15)$$

On the other hand, Eq. 14 has the qualitative merit of revealing that u_m^+ in the turbulent regime is equal to that in the laminar regime at the same value of a^+ less the integral term therein. The number of significant figures in the numerical coefficients of Eq. 15 and similar subsequent expressions can be justified in terms of representation of the computed values, but not in practical terms because of the unknown accuracy of the model from which these values are obtained.

Advantages of $\overline{(u'v')}^{++}$ over $\overline{(u'v')}^{+}$ as a correlating variable are rather slight in the formulations for flow in a round tube, which is why it was initially overlooked. However, the corresponding formulations for heat and mass transfer and for other geometries prove to be truly advantageous.

Mechanistic models

The Eddy-Viscosity Model. Boussinesq (1877) suggested the model

$$-\rho\overline{u'v'} = \mu_t \frac{du}{dy} \qquad (16)$$

where μ_t is an *eddy viscosity*, seemingly a purely artificial quantity. Introduction of this expression for $\rho\overline{u'v'}$ in Eq. 5 allows Eq. 6 to be reexpressed as

$$1 - \frac{y^+}{a^+} = \frac{du^+}{dy^+}\left(1 + \frac{\mu_t}{\mu}\right). \qquad (17)$$

According to Eq. 16 μ_t may be determined from local measurements of $\overline{u'v'}$ and u, while according to Eq. 17 local measurements of u are sufficient, although τ_w must also be determined. Elimination of du^+/dy^+ between Eqs. 7 and 17 leads to

$$\frac{\mu_t}{\mu} = \frac{\overline{(u'v')}^{++}}{1 - \overline{(u'v')}^{++}}. \qquad (18)$$

The one-to-one correspondence between μ_t/μ and $\overline{(u'v')}^{++}$, as revealed by Eq. 18, has surprising and important implications. Equation 18 indicates that the eddy viscosity is independent of the very diffusional mechanism in terms of which it was originally conceived. This result is a fortuitous consequence of the linearity of the differential momentum balance. Furthermore, μ_t/μ may be interpreted as simply the ratio of the turbulent shear stress to the viscous shear stress. The widespread criticism of the eddy viscosity because of its lack of a mechanistic or theoretical basis is thus misdirected. On the other hand, the possible merit of μ_t/μ relative to $(\overline{u'v'})^{++}$ as a correlating variable depends on whether or not it is a simpler or better behaved function of y^+ and a^+, whether or not it provides greater physical insight and whether or not it is more readily predictable from first principles. It fails in all of these respects. First, the generalized correlating equation for μ_t/μ devised by Churchill and Chan (1995a) from the same elements as used to construct Eq. 10 is more complicated (as can be inferred from Eq. 18). Second, the simple difference in the laminar and turbulent contributions to u^+ and u_m^+ as demonstrated by Eqs. 12 and 14, respectively, is not evident when Eq. 17 is integrated formally to obtain these two quantities. Third, neither the κ-ϵ model that has been proposed to predict μ_t/μ nor the κ-ϵ-$\overline{u'v'}$ model that has been proposed to predict $\overline{u'v'}$ directly, has any utility for round tubes. The empiricism of both of these models exceeds that involved in the construction of Eq. 10, the accuracy of their predictions is presumably less, and a significant amount of numerical computation is required to solve the additional partial differential equations.

Mixing-Length Model. Prandtl (1925) proposed, on the basis of a questionable analogy between the motion of turbulent eddies and the motion of the molecules of a gas, that

$$-\rho\overline{u'v'} = \rho l^2 \left(\frac{du}{dy}\right)^2. \qquad (19)$$

This expression may be combined with Eq. 5 to obtain, after reexpression in dimensionless form,

$$1-\frac{y^+}{a^+} = \frac{du^+}{dy^+} + \left(l^+ \frac{du^+}{dy^+}\right)^2. \qquad (20)$$

Elimination of du^+/dy^+ between Eqs. 7 and 20 yields

$$(l^+)^2 = \frac{(\overline{u'v'})^{++}}{\left(1-\frac{y^+}{a^+}\right)\left[1-(\overline{u'v'})^{++}\right]^2}. \qquad (21)$$

Equation 21 reveals that the mixing length is also independent of the specious mechanism by means of which it was conceived, again a fortuitous consequence of the linearity of the differential momentum balance. The mixing-length model therefore cannot be discredited on mechanistic grounds. The mixing-length model has generally been accorded more respect in the fluid mechanics community than the eddy-viscosity model. However, it is actually inferior in every respect, beginning with its more complicated relationship with $(\overline{u'v'})^{++}$ and thereby with measurable quantities. The historical roots of its unjustified prestige are apparent from the following examination of its initial numerical evaluation.

One of the most fateful plots in the history of fluid mechanics is reproduced in Figure 3, in which values of l, as determined by Nikuradse (1930) from measured values of

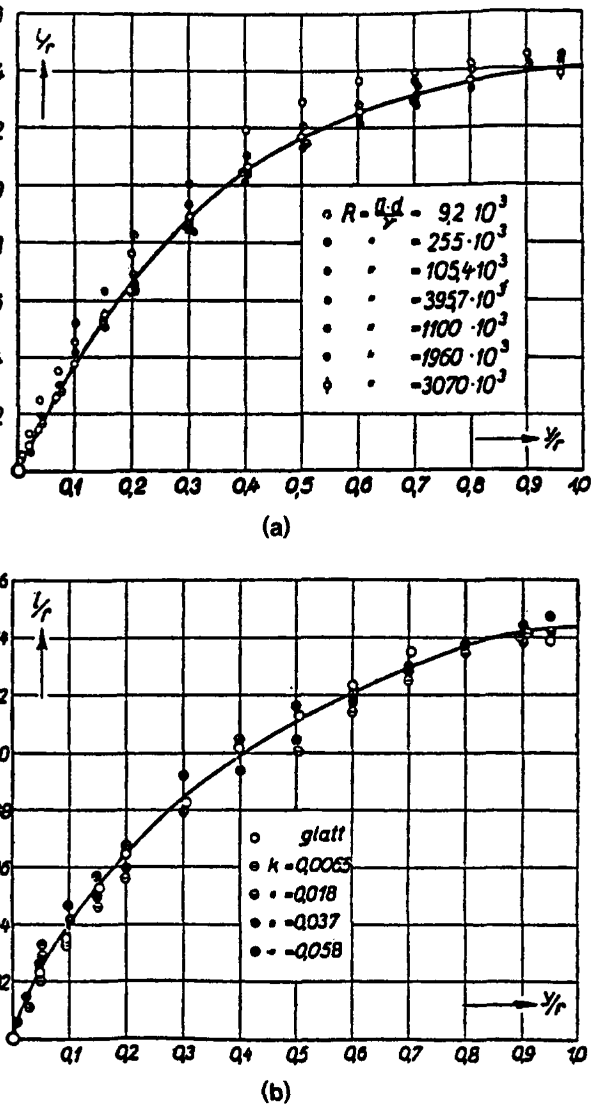

Figure 3. Mixing lengths as determined by Nikuradse (1930) in round tubes.

(a) Smooth tubes at large Reynolds numbers; (b) artificially roughened tubes at large Reynolds numbers.

$u\{y\}$, are plotted as l/a vs. y/a. Prandtl [see Nikuradse (1932)] concluded from this plot that near the center line l is independent of y, as he had expected, as well as of Re and a/e. Furthermore, although his original concept of an invariant l failed over the balance of the tube, he concluded that the variation near the wall could at least be expressed simply by the linear relationship

$$l = 0.4y. \qquad (22)$$

Both of these inferences are wrong. At the center line, l actually becomes unbounded and near the wall l^+ approaches equality to $(0.7)^{1/2}(y^+/10)^{3/2}$. These nuances were overlooked by Prandtl because of the gross scale of the abscissa of Figure 3, the excessive spacing of the measurements of $u\{y\}$, and their relative inaccuracy near the wall and near the center line. Equation 22 does provide a fair representation for 30 <

$y^+ < a^+/10$, but this relationship is also apparent from plots of $(\overline{u'v'})^{++}$ and μ_t/μ, and should not be credited uniquely to the mixing-length concept.

The mixing length is inferior to the eddy viscosity as a correlating variable because of its singular behavior at the center line as well as because of the greater complexity of Eqs. 20 and 21 relative to Eqs. 17 and 18. It is subject to the same failures in other flows as the eddy viscosity. Accordingly the mixing-length model is now of historical interest only.

Von Kármán (1930) proposed the expression

$$l = k' \left| \frac{du/dy}{d^2u/dy^2} \right| \tag{23}$$

for prediction of the mixing length, with k' implied to be the same coefficient as that of Eq. 22. This expression predicts the gross behavior displayed in Figure 3 reasonably well but shares the shortcomings of Eq. 23 for very small y and the inherent shortcoming of the mixing-length model itself at the center line.

Recapitulation

In a round tube the eddy viscosity and the mixing length are found to be independent of their mechanistic origin and therefore acceptable as correlating variables. However, they are both inferior in all respects to the dimensionless shear stress $(\overline{u'v'})^{++}$ itself. The κ-ϵ and κ-ϵ-$\overline{u'v'}$ models are applicable, but have no useful role in this geometry.

Equations 11 and 12 for u^+ and Eqs. 13 and 14 for u_m^+ in terms of $(\overline{u'v'})^{++}$ are the distilled result of the proposed new representation for momentum transfer. They are exact in the sense of being free of empiricisms and heuristics other than the postulate of the existence of fully developed flow. Of course the numerical evaluation of the integrals in these expressions introduces empiricism by virtue of the correlating expression used for $(\overline{u'v'})^{++}$.

Heat Transfer in a Round Tube

Because of the well-known close correspondence between momentum transfer and heat transfer, as illustrated by the many algebraic analogies that have been devised, the results of the preceding analysis of the models for momentum transfer might be expected to be adaptable with minimal modifications for heat transfer. This expectation is not fulfilled; the models for convective heat transfer prove to be inherently more complex.

The time-averaged partial differential equation for the conservation of energy in the y-direction in fully developed convection with negligible viscous dissipation in the fully developed turbulent flow in a round tube of a fluid with invariant physical properties may be combined with an overall energy balance to obtain

$$j = -k\frac{dT}{dy} + \rho c_p \overline{T'v'}. \tag{24}$$

Dedimensionalization by analogy to Eq. 6 results in

$$\frac{j}{j_w} = \frac{dT^+}{dy} + Pr(\overline{T'v'})^+, \tag{25}$$

while that corresponding to Eq. 7 yields the simpler result

$$\frac{j}{j_w}\left[1-(\overline{T'v'})^{++}\right] = \frac{dT^+}{dy^+}. \tag{26}$$

Here $T^+ \equiv k(T_w - T)(\tau_w \rho)^{1/2}/\mu j_w$, $(\overline{T'v'})^+ \equiv -k(T_w - T')v'\rho/\mu j_w$ and $(\overline{T'v'})^{++} \equiv \rho c_p \overline{T'v'}/j$. Alternative definitions of T^+ differing by a sign or by a factor of Pr have also appeared in the literature. The choice here was made to produce congruence insofar as possible with Eqs. 6 and 7. The definition of $(\overline{T'v'})^{++}$ was chosen in the interests of simplicity. From its definition, $(\overline{T'v'})^{++}$ may be noted to be the fraction of the local heat flux density in the y-direction due to turbulence and $1-(\overline{T'v'})^{++}$ the fraction due to thermal conduction.

In order to implement Eq. 26, correlative or predictive expressions are needed for j/j_w and $(\overline{T'v'})^{++}$. The former will be considered first.

Heat-flux density ratio

Insofar as viscous dissipation and the axial transport of energy by thermal conduction and the turbulent fluctuations are both negligible, the heat-flux density ratio may be expressed in terms of the following integral energy balance:

$$\frac{j}{j_w} = \frac{1}{R}\int_0^{R^2}\left(\frac{\partial T/\partial x}{\partial T_m/\partial x}\right)\left(\frac{u^+}{u_m^+}\right) dR^2. \tag{27}$$

Axial transport of energy by molecular and turbulent diffusion may become appreciable near the point of onset of heating for all rates of flow as $Pr \to 0$, but viscous dissipation is significant only for very high velocities and for very viscous liquids at high rates of shear. As contrasted with the shear stress ratio τ/τ_w, which is simply equal to R for fully developed flow, the heat-flux density ratio varies parametrically, by virtue of $(\partial T/\partial x)/(\partial T_m/\partial x)$, with a^+, Pr, the mode of heating on the wall, and the extent of the thermal development in the fluid stream. This parametric dependence of j/j_w is the principal source of complexitiy of the models and solutions for convection as compared to flow.

For the special case of a uniform heat-flux density on the wall and complete thermal development, the only conditions that are considered in detail herein, Eq. 27 reduces to

$$\frac{j}{j_w} = \frac{1}{R}\int_0^{R^2}\left(\frac{u^+}{u_m^+}\right) dR^2. \tag{28}$$

The heat-flux density ratio is seen to be a function only of the velocity field and thus independent of Pr and the thermal conditions. From Eq. 28 it is also evident that $j/j_w = \tau/\tau_w$ only for plug flow, a condition that is never attained physically. An exact expression from which j/j_w may be evaluated for any value of R and a^+ may be derived by introducing u^+ from Eq. 11 and u_m^+ from Eq. 28, and integrating by parts. The result may be expressed as

$$1+\gamma = M = \frac{j/j_w}{R} = \frac{1}{R^2} \times \left[\frac{\int_0^{R^4} \left[1-(\overline{u'v'})^{++}\right] dR^4 + 2R^2 \int_{R^2}^{1} \left[1-(\overline{u'v'})^{++}\right] dR^2}{\int_0^1 \left[1-(\overline{u'v'})^{++}\right] dR^4} \right]. \tag{29}$$

Rohsenow and Choi (1961) introduced the quantity $M = (j/j_w)/(\tau/\tau_w)$ in connection with Eq. 27, but did not evaluate or utilize it. Reichardt (1951) proposed the use of $\gamma = ((j/j_w)/R) - 1$ in derivations for heat transfer, and evaluated it from the equivalent of Eq. 27 for a uniform wall temperature, for which it depends on *Pr* as well as on *R* and a^+ or Re_D. Churchill and Balzhiser (1959) computed and presented graphical representations for j/j_w and $(j/j_w)R$ for a uniform wall temperature for various values of *Pr* and Re_D and degrees of thermal development as well as for uniform heating for various values of Re_D. Heng et al. (1977) have computed presumably more exact values of γ for uniform heating by using Eqs. 29 and 10. Their results may be represented with sufficient accuracy for all practical purposes by the following approximate expression in closed form:

$$\gamma = \frac{14Y\left(1+\frac{3}{2}Y-\frac{16}{7}Y^2\right) - \frac{60Y}{(1-Y)^2}\left[1-Y+(2-Y)\ln\{Y\}\right]}{49+60\ln\{a^+\}}. \tag{30}$$

The integration leading to Eq. 30 neglects the reduction of the velocity in the boundary layer near the wall corresponding to the first term on the righthand side of Eq. 10. The quantity γ, as predicted by Eqs. 29 and 30, ranges from zero at the wall to up to 0.266 at the center line for $a^+ = 150$ and from zero up to 0.106 for $a^+ = 10^6$.

A correlation for $(\overline{T'v'})^{++}$

A logical step at this point would be to examine experimental data and/or DNS values for $(\overline{T'v'})^{++}$, and if possible construct a generalized correlating equation analogous to Eq. 10 for $(\overline{u'v'})^{++}$. However, since the reliable experimental data and computed values by DNS for $\overline{T'v'}$ are essentially limited to $Pr = 0.7$ or 1.0, it proves to be advantageous to examine first the models involving the eddy conductivity and the correlations thereof.

The eddy-conductivity model

The analog of Eq. 16 for thermal transport is

$$\rho c\overline{T'v'} = -k_t \frac{dT}{dy}, \tag{31}$$

which may be combined with Eq. 25 to obtain, after dedimensionalization,

$$\frac{j}{j_w} = \frac{dT^+}{dy^+}\left(1+\frac{k_t}{k}\right). \tag{32}$$

Elimination of dT^+/dy^+ between Eqs. 26 and 32 results in

$$\frac{k_t}{k} = \frac{(\overline{T'v'})^{++}}{1-(\overline{T'v'})^{++}}. \tag{33}$$

The one-to-one correspondence between k_t/k and $(\overline{T'v'})^{++}$ indicates that the eddy conductivity is also independent of the diffusional mechanism in terms of which it was originally conceived. Furthermore, Eq. 33 indicates that k_t/k may be interpreted simply as the ratio of the heat-flux density due to turbulence to that due to molecular conduction. Of course, as might be expected, the correlations for k_t/k are just as limited as those for $(\overline{T'v'})^{++}$ since they are based on the same information. In response to this diffuculty, Eq. 33 is ordinarily expanded as

$$\frac{j}{j_w} = \frac{dT^+}{dy^+}\left[1+\left(\frac{c_p\mu}{k}\right)\left(\frac{k_t}{c_p\mu_t}\right)\frac{\mu_t}{\mu}\right] \tag{34}$$

and then rewritten as

$$\frac{j}{j_w} = \frac{dT^+}{dy^+}\left[1+\left(\frac{Pr}{Pr_t}\right)\frac{\mu_t}{\mu}\right], \tag{35}$$

where $Pr_t = c_p\mu_t/k_t$ is called the *turbulent Prandtl number.* The task of developing a correlating equation for k_t/k is thus replaced by the task of developing separate ones for Pr_t and μ_t. The advantage of this widely used approach is that correlating equations for μ_t are reasonably well established (for example, the combination of Eqs. 10 and 18 provides one), while Pr_t is found to vary only slightly from unity for most conditions. Indeed, many early workers simply postulated a value of unity for Pr_t. Expressions for the prediction of Pr_t are examined in the succeeding section.

Elimination of dT^+/dy^+ between Eqs. 26 and 35, followed by elimination of μ_t/μ by means of Eq. 18, leads to the somewhat surprising result that

$$\frac{Pr_t}{Pr} = \left(\frac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)\left(\frac{1-(\overline{T'v'})^{++}}{(\overline{T'v'})^{++}}\right). \tag{36}$$

Thus Pr_t, just as μ_t and k_t, is independent of its origin in a diffusional mechanism and may be interpreted in general physical terms as the ratio of the shear stress (or momentum-flux density) due to turbulence to that due to viscosity, divided by the corresponding ratio for the heat-flux density. This establishment of a physical character for Pr_t suggests the following alternative to Eq. 26:

$$\frac{j}{j_w} = \frac{dT^+}{dy^+}\left(1+\frac{Pr}{Pr_t}\left[\frac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right]\right). \tag{37}$$

The requirement of a correlative or predictive expression for $(T'v')^{++}$ in Eq. 25 is thus replaced by the requirement of one for Pr/Pr_t. As indicated in connection with Eq. 35, this is an improvement since an accurate correlating equation is available for $(\overline{u'v'})^{++}$, and since the variance of Pr_t is relatively constrained for $Pr \geq 0.7$.

One further alternative representation is worthy of examination. It begins with the expansion of Eq. 32 as

$$\frac{j}{j_w} = \frac{dT^+}{dy^+}\left[\left(\frac{k+k_t}{c_p(\mu+\mu_t)}\right)\left(\frac{c_p\mu}{k}\right)\left(\frac{\mu+\mu_t}{\mu}\right)\right], \quad (38)$$

which may be rewritten as

$$\frac{j}{j_w} = \frac{dT^+}{dy^+}\left(\frac{Pr}{Pr_T}\right)\frac{\mu_T}{\mu} \quad (39)$$

where $Pr_T = c_p(\mu+\mu_t)/(k+k_t)$ is the *total Prandtl number* and $\mu_T = \mu + \mu_t$ is the *total viscosity*. From Eqs. 18, 36, 37, and 39 it follows that

$$\frac{Pr_T}{Pr} = \frac{1-(\overline{T'v'})^{++}}{1-(\overline{u'v'})^{++}}. \quad (40)$$

In view of Eqs. 33 and 36, this demonstration of a physical significance for Pr_T, independent of its diffusional mechanistic origin, is hardly surprising. The ratio Pr_T/Pr is seen from Eq. 40 to correspond to the ratio of the fraction of the heat-flux density due to thermal conduction to the fraction of the momentum-flux density (the shear stress) due to viscosity. As a consequence of the physical validation of Pr_T, Eq. 39 with μ_T/μ replaced by $1/(1-(u'v')^{++})$, namely

$$\frac{j}{j_w}\left[1-(\overline{u'v'})^{++}\right]\left(\frac{Pr_T}{Pr}\right) = \frac{dT^+}{dy^+} \quad (41)$$

is a viable alternative to Eq. 37. Based on their definitions,

$$\frac{1}{Pr_T} = \frac{(\overline{u'v'})^{++}}{Pr_t} + \frac{1-(\overline{u'v'})^{++}}{Pr}. \quad (42)$$

Since substitution for Pr_T in Eq. 41 from Eq. 42 results in Eq. 37, they would appear to be wholly equivalent. However, one or the other may actually be preferable for integration in particular regimes of y^+, a^+, and Pr owing to the relative invariance of Pr_t and Pr_T. The diffusional mechanism of turbulent transport, which has disappeared in Eqs. 5–7, 24–25, and 41, appears to have resurfaced in Eq. 37. In view of the relationship between Eqs. 37 and 41 mentioned earlier, this is not a deficiency of the former: rather it may be interpreted as providing some belated credit for the intuition of Boussinesq in postulating Eq. 16.

From Eq. 42, it may be inferred that $Pr_T \to Pr_t/(\overline{u'v'})^{++}$ for $Pr \to \infty$ and finite values of $(\overline{u'v'})^{++}$, that $Pr_T \to Pr/[1-(\overline{u'v'})^{++}]$ for $Pr \to 0$ and finite values of $1-(\overline{u'v'})^{++}$, and that $Pr_T = Pr_t$ for $Pr = Pr_t$, a condition that occurs experimentally for $Pr \cong 0.86$.

Predictive and correlative expression for Pr_t

The primary role of Eqs. 37 and 41 is to provide predictions, either analytical or numerical, for $T^+\{y^+, a^+, Pr\}$ and $Nu_D\{a^+, Pr\}$. Since $(\overline{u'v'})^{++}$ and j/j_w are predictable with good accuracy by means of Eqs. 10 and 29 or 30, respectively, the critical factor in the use of Eqs. 37 and 41 is the accuracy of the predictions for Pr_t and Pr_T. In view of Eq. 42, a prediction of Pr_t also provides a reasonably accurate one for Pr_T and vice versa, except possibly in the limit of $y^+ \to 0$ and thereby $(\overline{u'v'})^{++} \to 0$. Since values of Pr_T or Pr_t are the weakest link in the application of Eqs. 37 and 41 or of any of their classic predecessors, the determination of Pr_t or Pr_T from experimental data and DNS, and the reliability and generality of correlations and predictive expressions will be examined in some detail.

The turbulent Prandtl number has generally been postulated explicitly or implicitly to be a function only of $(\overline{u'v'})^{++}$ or μ_t/μ and Pr, and thereby independent of geometry and the thermal boundary conditions. This postulate is of great utility since it allows experimental measurements for parallel plates, for which the variation of the heat flux density may be avoided, and results from DNS, which are yet confined to parallel plates, to be interpreted as directly applicable for round tubes.

The experimental determination of Pr_t from the combination of Eqs. 17 and 32 is subject to considerable inaccuracy because of the necessity of determining j/j_w and of differentiating the velocity and temperature measurements. The accuracy of the determinations of μ_t and k_t is particularly poor near the center line owing to the approach of both τ/τ_w and du/dy and of both j/j_w and dT/dy to zero, and even poorer near the wall owing to the substraction of the molecular contributions to the momentum- and heat-flux densities from τ/τ_w and j/j_w, respectively, to obtain small differences from relatively large values. These inaccuracies are greatly increased by taking the ratio of the so-determined values of μ_t and k_t. This process is perhaps best illustrated by the determinations of Abbrecht and Churchill (1960), which, despite their age, are among the most accurate values ever obtained by this procedure. They measured the velocity and the temperature at the same points within an air stream by hot-wire anemometry, the heat-flux density at the wall with a calorimeter, and determined j/j_w by means of measurements of the longitudinal profile of the time-averaged temperature. They then differentiated the velocities and temperatures graphically using equal-area plots. Their independent measurements of μ_t and k_t are in excellent agreement with prior and subsequent determinations and theoretically based asymptotes. Although their resulting values of Pr_t appear to be valid within the turbulent core near the wall, they are highly scattered near the center line and very near the wall, and indeed appear to predict a completely erroneous trend at the wall (downward instead of upward). One unique and lasting contribution of these determinations, which were for a step increase in wall-temperature, is the finding that Pr_t, is independent of the thermal development. The good agreement of these values of Pr_t with those determined by Page et al. (1952) for flow between parallel plates at different uniform temperatures (resulting in $j/j_w = 1$) confirms their independence from geometry as well as from the thermal boundary condition, at least in this instance.

Although a number of sets of seemingly accurate experimental measurements of $\overline{u'v'}$ and $\overline{T'v'}$ in air flowing between parallel plates have been carried out in recent years, μ_t and k_t have in each instance apparently been determined from the equivalent of Eqs. 17 and 31, respectively, rather than more directly from Eqs. 18 and 33. In any event these determinations become highly inaccurate near the wall and near the central plane. Equation 36, which is subject to the same inaccuracies, does not appear to have been utilized directly.

The experimental determinations of μ_t, k_t, and Pr_t mentioned earlier are all for air. The corresponding measurements are much more difficult for liquid metals (low-Prandtl-number fluids) for obvious reasons, and for viscous liquids (high-Prandtl-number fluids) because the entire development of the temperature field occurs very near the wall, thereby requiring the use of low overall temperature differences to avoid significant variations in μ with T.

Direct numerical simulations would appear to be a means of avoiding the uncertainties in Pr_t as determined experimentally. However, the results to date are somewhat limited and disappointing. First, all of these simulations, with one exception have been for $Pr = 0.7$, 0.72, or 1.0, and second, with only a few exceptions, values have been presented graphically for $\overline{u'v'}$, $\overline{T'v'}$, $\overline{u'v'}/\overline{T'v'}$, μ_t, and/or k_t, but not for Pr_t. The latter omission is presumably because of the uncertainty in that quantity. Lyons et al. (1991) did present a curve representing values of Pr_t that increased from about 0.75 at the center plane to about unity at the wall. This curve, which is for $Pr = 1$ and parallel plates at different uniform temperatures, wobbles somewhat, suggesting that even the values determined by DNS may be of borderline precision for this purpose. Kasagi et al. (1992) presented a curve that differs only slightly from that of Lyons et al., despite a value of $Pr = 0.72$ and equal uniform heating of both walls. The Lagrangian direct numerical simulations of Papavassiliou and Hanratty (1997) constitute the exception noted previously with respect to values of Pr. Although they do not present specific values for Pr_t, their graphical results for $k_t/\rho c_p$ fall increasingly below μ_t/ρ as Pr increases from 1 to 2,400 and y^+ decreases below 5.

Many empirical and semitheoretical correlating equations have been proposed for Pr_t. None of them appear to be totally reliable or general. Three of them may be cited as representative. Experimental values of Pr_t in the turbulent core for $Pr \geq 0.7$ have been correlated by Jischa and Rieke (1979) and others with expressions such as

$$Pr_t = 0.85 + \frac{0.015}{Pr}. \quad (43)$$

Over the purported range of validity of Eq. 43, Pr_t varies only from 0.87 to 0.85. An expression with a broader purported range of y^+ and a^+, as wholly represented by $\mu_t/\mu \equiv (\overline{u'v'})^{++}/[1-(\overline{u'v'})^{++}]$, is that of Notter and Sleicher (1972):

$$Pr_t = \frac{1 + 90 Pr^{3/2}(\mu_t/\mu)^{1/4}}{\left(1 + \frac{10}{35 + (\mu_t/\mu)}\right)\left[0.025 Pr(\mu_t/\mu) + 90 Pr^{3/2}(\mu_t/\mu)^{1/4}\right]}. \quad (44)$$

Equation 44 is in reasonable accord with Eq. 43 and with the computed values of Lyons et al. for $Pr = 1$, but of course was not designed to predict a sharp increase for large Pr and small μ_t/μ in accord with the recent computed values of Papavassiliou and Hanratty. Yahkot et al. (1987) derived the following expression for the prediction of the total Prandtl number by *renormalization group theory*:

$$\left(\frac{1.1793 - \frac{1}{Pr_T}}{1.1793 - \frac{1}{Pr}}\right)^{0.65}\left(\frac{2.1793 - \frac{1}{Pr_T}}{2.1793 - \frac{1}{Pr}}\right)^{0.35} = 1 - (\overline{u'v'})^{++}. \quad (45)$$

Equation 45, which is asserted by the authors to be free of any "experimentally adjustable parameters," is attractive because of its simplicity and purported generality. However, the comparisons by Yahkot et al. of values of T^+ and Nu_D based on Eq. 45 with experimental values do not constitute a critical test and the predicted values of Pr_t for large Pr decrease as $y^+ \to 0$ [and $(\overline{u'v'})^{++} \to 0$], as contrasted with the increase computed by Papavassiliou and Hanratty.

It appears that a reliable and comprehensive expression for the prediction of Pr_t does not yet exist. Until one is established, Eq. 45 is recommended, except for $y^+ \to 0$ for $Pr > 1$, for which Eq. 44 is suggested. Yahkot et al. (1987) imply that Eq. 34 is applicable for all thermal boundary conditions as well as all flows. As noted earlier, Abbrecht and Churchill (1960) provided support far in advance for that implication, in that their experimentally determined values of Pr_t were found to be independent of axial distance, that is of the degree of thermal development following a step in wall temperature in fully developed flow in a round tube.

Implementation

In order to evaluate $T^+\{y^+, a^+, Pr\}$ and $Nu_D\{a^+, Pr\}$ using Eq. 41, it is convenient to substitute $(1+\gamma)R$ for j/j_w, per Eq. 29, and then $-a^+dR$ for dy^+, thereby obtaining

$$-a^+(1+\gamma)\left[1-(\overline{u'v'})^{++}\right]\left(\frac{Pr_T}{Pr}\right)R\,dR = dT^+, \quad (46)$$

which, upon formal integration, gives

$$T^+ = \frac{a^+}{2}\int_{R^2}^{1}(1+\gamma)\left[1-(\overline{u'v'})^{++}\right]\left(\frac{Pr_T}{Pr}\right)dR^2. \quad (47)$$

By definition

$$T_m^- = \int_0^1 T^+\left(\frac{u^+}{u_m^+}\right)dR^2. \quad (48)$$

Substituting for T^+ from Eq. 47 and then integrating by parts leads, by virtue of Eq. 28, to

$$T_m^+ = \frac{a^+}{4}\int_0^1(1+\gamma)^2\left[1-(\overline{u'v'})^{++}\right]\left(\frac{Pr_T}{Pr}\right)dR^4. \quad (49)$$

Although Eqs. 47 and 49 are applicable for fully developed laminar flow and convection with $\overline{(u'v')}^{++}$ set to zero and Pr_T/Pr to unity, the additivity that was observed in Eqs. 12 and 14 does not prevail for turbulent convection because γ depends on the velocity distribution. Since the definition of T_m^+ may be expanded as

$$T_m^+ = \left(\frac{k(T_w - T_m)}{aj_w}\right)\left(\frac{a(\tau_w \rho)^{1/2}}{\mu}\right) = \frac{2a^+}{Nu_D}, \tag{50}$$

it follows from Eq. 49 that

$$Nu_D = 8/\int_0^1 (1+\gamma)^2 \left[1 - \overline{(u'v')}^{++}\right]\left(\frac{Pr_T}{Pr}\right) dR^4. \tag{51}$$

If Eq. 37 had been used rather than Eq. 41 as the starting point, the following alternative expression would have been obtained:

$$Nu_D = 8/\int_0^1 \frac{(1+\gamma)^2 dR^4}{1 + \left(\frac{Pr}{Pr_t}\right)\left(\frac{\overline{(u'v')}^{++}}{1 - \overline{(u'v')}^{++}}\right)}. \tag{52}$$

Equations 51 and 52 are more convenient as the starting point for the derivation of analytical expressions for Nu_D, and presumably result in more accurate numerical integrations than prior expressions in terms of u^+/u_m^+ and $(\mu_t/\mu)(Pr/Pr_t)$ that involve double or triple integrations. The closest previous expression is that of Lyon (1951), who derived the equivalent of Eq. 52 with $1/R\int_0^{R^2} (u^+/u_m^+)\, dR^2$ substituted for $1+\gamma$ and k_T/k for the entire denominator.

For $Pr \to 0$, Eqs. 51 and 52 both reduce to

$$Nu_D\{0\} = 8/\int_0^1 (1+\gamma)^2\, dR^4 = \frac{8}{(1+\gamma)_m^2}, \tag{53}$$

which serves as exact lower bound for Nu_D for the turbulent regime. Values corresponding to Eq. 53 have been computed by a number of investigators in the past (for example, Kays and Crawford, 1980, Table 13.1). Values of $(1+\gamma)_m^2$ and $Nu_D\{Pr = 0\}$, of presumably greater accuracy have recently been computed by Heng et al. (1997) using Eqs. 10 and 29.

For $Pr = Pr_t = Pr_T$, Eq. 51 may be integrated formally and combined with Eq. 13 to obtain

$$Nu_D\{1\} = \frac{Re_D f/2}{(1+\gamma)_{wm}^2}, \tag{54}$$

where $(1+\gamma)_{wm}^2$ is the integrated-mean value weighted by $1 - \overline{(u'v')}^{++}$ over R^4 from 0 to 1. Values of $(1+\gamma)_{wm}^2$, which represents the deviation from the Reynolds analogy in a uniformly heated round tube, are also given by Heng et al. Their results indicate small but significant deviations for all values of Re_D.

Insofar as the variation of Pr_t with y^+ may be neglected, the following asymptotic and limiting solutions for $Pr \to \infty$ may be derived from Eq. 52:

$$Nu_D\{\infty\} = 0.07343\left(1 - \frac{Pr_t}{Pr}\right)^{4/3}\left(\frac{Pr}{Pr_t}\right)^{1/3} Re_D\left(\frac{f}{2}\right)^{1/2}$$

$$\to 0.07343\left(\frac{Pr}{Pr_t}\right)^{1/3} Re_D\left(\frac{f}{2}\right)^{1/2}. \tag{55}$$

The details of the derivation of Eq. 55 are given by Churchill (1996). Although this expression, with $Pr_t \cong 0.85$, appears to be in good agreement with experimental data for heat transfer [see, for example, Churchill (1977)], the prediction by Papavassiliou and Hanratty (1997), by means of Lagrangian direct numerical simulations, of a continual decrease in Pr_t as $y^+ \to 0$ for large values of Pr raises some question as to the postulate of an invariant value in the derivation of Eq. 55 and thereby of its range of validity if any.

The several expressions for Nu_D that imply a value of γ from Eqs. 29 or 30 are obviously limited to a uniformly heated wall. The corresponding expressions for a uniform wall temperature will not be derived herein since they relate only indirectly to the primary subject matter of this article, namely the models for the turbulent transport of energy. The differences in Nu_D are negligible for $Pr \geq 0.7$, but are significant for smaller Pr as indicated by the plot of computed values in figure 13.5 of Kays and Crawford (1980).

As a final note before turning to other geometries, it sould be emphasized that Eqs. 47, 49, and 51 for T^+, T_m^+, and Nu_D, respectively, as well as Eq. 29 for γ, Eq. 36 for Pr_t, and Eq. 40 for Pr_T are exact relationships in that they invoke no empiricism or heuristic postulates other than the existence of fully developed flow and convection.

Other Geometries and Conditions

The simplifications and improvements described earlier for the representation of fully developed momentum and energy transfer in a uniformly heated round tube were achieved by using $\overline{(u'v')}^{++}$ and γ rather than μ_t/μ and u/u_m as primary variables. Comparable advantages appear to accrue from the use of these new variables in other flows and for other thermal boundary conditions. However, the extended applications must be devised case by case. In the examples that follow, this process becomes progressively more difficult because of geometrical complexities.

Identical parallel plates

According to the analogy of MacLeod (1951), which appears to be confirmed within the uncertainty of the relevant experimental data, $\overline{(u'v')}^{++}\{y^+, b^+\}$ is identical to $\overline{(u'v')}^{++}\{y^+, a^+\}$. Hence Eq. 10 is applicable for flow between parallel plates if a^+ is replaced by b^+, but j/j_w differs as illustrated below. Accordingly, the construction of the new formulations for u_m^+, T^+, and Nu is quite straightforward.

First, integrating u^+, as given by Eq. 11, but in terms of $Z = 1 - (y^+/b^+)$ rather than $R = 1 - (y^+/a^+)$, over the planar cross-section gives

$$u_m^+ = \frac{b+}{3}\int_0^1 \left[1-\overline{(u'v')}^{++}\right] dZ^3 = \frac{b^3}{3}\left[1-\int_0^1 \overline{(u'v')}^{++}\, dZ^3\right] \tag{56}$$

as compared to Eqs. 13 and 14. Values of u_m^+ computed from Eq. 56 with $\overline{(u'v')}^{++}$ from Eq. 10 were found by Churchill and Chan (1994) to be represented almost exactly by the semitheoretical expression

$$u_m^+ = 3.3618 - \frac{190.83}{b^+} + 2.5\ln\{b^+\}. \tag{57}$$

The presence of a term in $(a^+)^2$ in Eq. 15 and the absence of one in $(b^+)^2$ in Eq. 57 indicates that the equivalent-diameter concept is not valid with respect to the friction factor. This conclusion may be inferred to apply to all geometries.

For equal uniform heating of the fluid from both plates, an integral energy balance gives

$$\frac{j}{j_w} = \int_0^Z \left(\frac{u^+}{u_m^+}\right) dZ \tag{58}$$

as compared to Eq. 28 for a round tube. Since $\tau/\tau_w = 1 - y^+/b^+ = Z$, it follows that

$$1+\gamma = \frac{1}{Z}\int_0^Z \left(\frac{u^+}{u_m^+}\right) dZ. \tag{59}$$

Substituting u^+ from Eq. 11 and u_m^+ from Eq. 56 leads to

$$1+\gamma = \frac{\frac{1}{Z}\int_0^{Z^3}\left[1-\overline{(u'v')}^{+}\right] dZ^3 + \frac{3}{2}\int_{Z^3}^1\left[1-\overline{u'v'}^{+}\right] dZ^2}{\int_0^1\left[1-\overline{u'v'}^{++}\right] dZ^3}. \tag{60}$$

Values of γ computed using Eqs. 60 and 10 are represented with sufficient accuracy for all practical purposes by

$$\gamma = \frac{5(1-Z)(3-2Z)Z^2 - 30\ln\{1-Z\}}{Z(41+30\ln\{b^+\})}. \tag{61}$$

They range from zero at the wall to 0.186 at the central plane for $b^+ = 150$, and from zero to 0.065 for $b^+ = 10^6$.

Equation 47 is applicable for the temperature distribution between the plates with b^+ in place of a^+, Z in place of R, and γ from Eq. 60 or 61 instead of from Eq. 29 or 30. However,

$$T_m^+ = \int_0^1 T^+\left(\frac{u^+}{u_m^+}\right) dZ, \tag{62}$$

which leads to

$$T_m^+ = \frac{b^+}{3}\int_0^1 (1+\gamma)^2\left[1-\overline{(u'v')}^{++}\right]\left(\frac{Pr_T}{Pr}\right) dZ^3 \tag{63}$$

and

$$Nu_D = 12/\int_0^1 (1+\gamma)^2\left[1-\left(\overline{u'v'}^{++}\right)\right]\left(\frac{Pr_T}{Pr}\right) dZ^3. \tag{64}$$

For $Pr \to 0$, for which $[1-\overline{(u'v')}^{++}]Pr_T/Pr \to 1$, Eq. 64 reduces to

$$Nu_D\{0\} \to \frac{12}{(1+\gamma^2)_m}, \tag{65}$$

where here $(1+\gamma)_m^2$ is the integrated-mean value of $(1+\gamma)^2$ with respect to Z^3. For $Pr = Pr_T = Pr_t$, Eq. 64 reduces to

$$Nu_D\{1\} \to 12/\int_0^1 (1+\gamma^2)\left[1-\overline{(u'v')}^{++}\right] dZ^3 = \frac{Ref/2}{(1+\gamma)_{wm}^2}, \tag{66}$$

where here $(1+\gamma)_{wm}^2$ is the integrated-mean value with respect to Z^3, weighted by $1-\overline{(u'v')}^{++}$. This quantity is seen to constitute a different correction for the Reynolds analogy. Equation 55 is applicable without modification for $Pr \to \infty$.

For parallel plates with different uniform temperatures, the heat-flux density is invariant along the wall and across the channel, that is, $j/j_w = 1$. This special condition has often been utilized experimentally in the laboratory and in direct numerical simulations because of its structural simplicity. For this process

$$T^+ = b^+\int_0^Y\left[1-\overline{(u'v')}^{++}\right]\left(\frac{Pr_T}{Pr}\right) dY \tag{67}$$

$$T_c^+ = b^+\int_0^1\left[1-\overline{(u'v')}^{++}\right]\left(\frac{Pr_T}{Pr}\right) dY \tag{68}$$

and

$$Nu_{2b} = \frac{2b^+}{2T_c^+} = \frac{1}{\int_0^1\left[1-\overline{(u'v')}^{++}\right]\left(\frac{Pr_T}{Pr}\right) dY}. \tag{69}$$

The Nusselt number Nu_{2b} represents the rate of heat transfer from one wall to the other and hence is based on the total temperature difference $2T_c^+$ and the distance $2b^+$ between the plates. In the limit of $Pr \to 0$, Nu_{2b} simply reduces to unity, but for $Pr = Pr_T = Pr_t$,

$$Nu_{2b}\{1\} = \frac{1}{\int_0^1\left[1-\overline{(u'v')}^{++}\right]dY} = \frac{1}{\left[1-\overline{(u'v')}^{++}\right]_m}. \tag{70}$$

Values of $Nu_{2b}\{1\}$, as computed from Eq. 70, greatly exceed unity. For $Pr \to \infty$, Eq. 55 is applicable in terms of half the temperature difference, and is of course independent of the characteristic dimension.

A general solution and limiting forms may also readily be derived for uniform heating on only one wall.

Planar Couette flow

Relatively simple formulations in terms of $(\overline{u'v'})^{++}$ and Pr_T are possible for the turbulent regime of the somewhat idealized flow generated by one plate moving parallel to a fixed plate. A correlating equation for $(\overline{u'v'})^{++}$ analogous to Eq. 10 may readily be constructed on the basis of antisymmetry and the universal behavior near walls. The attractive feature of this flow with respect to analysis is that the shear stress is uniform across the channel. It follows that

$$u^+ = b^+ \int_0^Y \left[1-(\overline{u'v'})^{++}\right] dY \qquad \text{for} \qquad 0 \le Y \le 1 \quad (71)$$

and

$$u_c^+ = u_m^+ = \frac{u_w^+}{2} = b^+ \int_0^1 \left[1-(\overline{u'v'})^{++}\right] dY. \quad (72)$$

The velocity distribution for $1 \le Y \le 2$ follows from antisymmetry.

For different uniform temperatures on the two walls, $j/j_w = 1$ and Eqs. 67–69, as well as the special cases of Eq. 69 for $Pr \to 0$, $Pr = Pr_T = Pr_t$, and $Pr \to \infty$, are directly applicable although, of course, $(\overline{u'v'})^{++}\{y^+, b^+\}$ is different. Simple solutions are also feasible for some other thermal boundary conditions, for example, for uniform equal heating, although in this instance an expression must be derived for the variation of j/j_w with Y.

Flow through a circular annulus

This geometry is of great importance because of the widespread use of double-pipe heat exchanges in which the outer passage consists of a concentric circular annulus heated more or less uniformly on the inner surface and effectively insulated on the outer surface. The time-averaged differential force-momentum balance for this flow may be written in terms of the new variable $(\overline{u'v'})^{++}$ as follows:

$$\frac{\tau}{\tau_{w1}}\left[1-(\overline{u'v'})^{++}\right] = \frac{du^+}{dr^+}, \quad (73)$$

where here, τ_{w1}, the shear stress on the inner wall may be implied to be utilized in defining u^+ and r^+. Equation 73 has the same form as Eq. 9, but three complications arise in its integration to determine u^+ and u_m^+. First, the behavior depends upon the aspect ratio a_1/a_2, where a_1 and a_2 are the radii of the inner and outer surfaces, respectively. Second, the shear stress distribution within the fluid is given by the nonlinear relationship

$$\frac{\tau}{\tau_{w1}} = \left(\frac{a_0^2 - r^2}{a_0^2 - a_1^2}\right)\frac{a_1}{r} = \left(\frac{\left(\frac{a_0}{a_1}\right)^2 - R_1^2}{\left(\frac{a_0}{a_1}\right)^2 - 1}\right)\frac{1}{R_1}, \quad (74)$$

where $R_1 = r/a_1$ and a_0 is the radius at which the shear stress goes to zero. Third, the value of a_0 in turbulent flow differs from that in laminar flow and from a_{max}, the value of the maximum in the velocity distribution and is unknown *a priori*.

Despite these difficulties, formal and potentially useful expressions for a^+, u_m^+, T^+, and Nu_D in terms of $(\overline{u'v'})^{++}$ and Pr_T or Pr_t may be constructed as follows. Integrating Eq. 73 from $u^+ = 0$ at $R_1 = 1$ gives

$$u^+ = a_1^+ \int_1^{R_1}\left(\frac{\tau}{\tau_{w1}}\right)\left[1-(\overline{u'v'})^{++}\right], dR_1. \quad (75)$$

For an annulus

$$u_m^+ = \frac{1}{\left[\left(\frac{a_2}{a_1}\right)^2 - 1\right]}\int_1^{(a_2/a_1)^2} u^+ \, dR_1^2. \quad (76)$$

Substituting u^+ from Eq.75 in Eq. 76 and integrating by parts then gives

$$u_m^+ = \frac{a_1^+}{\left[\left(\frac{a_2}{a_1}\right)^2 - 1\right]}\int_1^{a_2/a_1}\left(\frac{\tau}{\tau_{w1}}\right)\left[1-(\overline{u'v'})^{++}\right] \times \left[\left(\frac{a_2}{a_1}\right)^2 - R_1^2\right] dR_1. \quad (77)$$

Equations 76 and 77, as well as Eq. 74, which provides the dependence of τ on R, are exact, but empirical correlating equations are required for $(\overline{u'v'})^{++}$ and a_0.

Rehme (1974) correlated his own experimental data for a_0 as well as that of others at $Re_D \equiv 2(a_2 - a_1)u_m\,\rho/\mu \cong 10^5$ with the empirical equation

$$\frac{a_0 - a_1}{a_2 - a_0} = \left(\frac{a_1}{a_2}\right)^{0.386}. \quad (78)$$

The accuracy of Eq. 78, particularly for other values of Re_D is uncertain.

A generalized correlating equation for $(\overline{u'v'})^{++}$, such as Eq. 10 for round tubes, has not yet been developed for annuli but appears to be feasible. Using a κ-ϵ-$\overline{u'v'}$ model, Hanjalić and Launder (1972b) have predicted values of $\overline{u'v'}$ and a_0 for the single condition of $Re_D = 2.4 \times 10^5$ and $a_1/a_2 = 0.088$ that are in good agreement with experimental data. This is a possible alternative to the use of a correlating equation for $(\overline{u'v'})^{++}$, but involves a significant amount of computing.

Further detail concerning the relationships for turbulent flow in an annulus, including expressions for the friction factors of the two surfaces as well as for the overall one may be found in Churchill and Chan (1995b).

The time-averaged differential energy balance for an annulus with an imposed uniform heat flux density j_{w1} on the inner surface and an outer insulated surface may be expressed as follows:

$$\frac{j}{j_{w1}}\left[1-(\overline{u'v'})^{++}\right]\left(\frac{Pr_T}{Pr}\right)=\frac{dT^+}{dr^+}, \tag{79}$$

with

$$\frac{j}{j_{w1}}=\frac{1}{\left[\left(\frac{a_2}{a_1}\right)^2-1\right]R_1}\int_{R_1^2}^{(a_2/a_1)^2}\left(\frac{u^+}{u_m^+}\right)dR_1^2. \tag{80}$$

Substituting u^+ in Eq. 80 from Eq. 75 and integrating by parts leads to the following expression in terms of $(\overline{u'v'})^{++}$:

$$\frac{j}{j_{w1}}=\frac{a_1^+}{u_m^+R_1}\left[\left(\frac{\left(\frac{a_2}{a_1}\right)^2-R_1^2}{\left(\frac{a_2}{a_1}\right)^2-1}\right)\int_1^R\left(\frac{\tau}{\tau_{w1}}\right)\left[1-(\overline{u'v'})^{++}\right]dR_1 + \int_{R_1}^{a_2/a_1}\left(\frac{\left(\frac{a_2}{a_1}\right)^2-R_1^2}{\left(\frac{a_2}{a_1}\right)^2-1}\right)\left(\frac{\tau}{\tau_{w1}}\right)\left[1-(\overline{u'v'})^{++}\right]dR_1\right]. \tag{81}$$

Integration of Eq. 79 from the inner heated surface at $R_1=1$, where $T^+=0$, gives

$$T^+=a_1^+\int_1^{R_1}\left(\frac{j}{j_{w1}}\right)\left[1-(\overline{u'v'})^{++}\right]\left(\frac{Pr_T}{Pr}\right)dR_1, \tag{82}$$

which may be substituted in

$$T_m^+=\frac{1}{\left[\left(\frac{a_2}{a_1}\right)^2-1\right]}\int_1^{(a_2/a_1)^2}T^+\left(\frac{u^+}{u_m^+}\right)dR_1^2 \tag{83}$$

to yield after integration by parts and by virtue of Eq. 80,

$$T_m^+=\frac{2a^+}{3}\int_1^{(a_2/a_1)^2}\left(\frac{j}{j_{w1}}\right)^2\left[1-(\overline{u'v'})^{++}\right]\left(\frac{Pr_T}{Pr}\right)dR_1^2. \tag{84}$$

It follows that

$$Nu_D=\left(\frac{2a_1^+}{T_m^+}\right)\left(\frac{a_2}{a_1}-1\right)=\frac{4\left(\frac{a_2}{a_1}-1\right)}{\int_1^{(a_2/a_1)^2}\left(\frac{j}{j_{w1}}\right)^2\left[1-(\overline{u'v'})^{++}\right]\left(\frac{Pr_T}{Pr}\right)dR_1^2}. \tag{85}$$

The introduction of $(1+\gamma)(\tau/\tau_{w1})$ for (j/j_{w1}) in Eqs. 82, 84, and 85 is not particularly helpful in view of the complexity of the relationship given by Eq. 74. Equations 79–85 are all exact, although of course empiricism is introduced in numerical evaluations by the correlating equations for $(\overline{u'v'})^{++}$, a_0/a_1, and Pr_T or Pr_t. The complexity of the expressions for T^+, T_m^+, and Nu_D relative to their counterparts for a round tube arises wholly from the expressions for τ_w/τ_{w1} and j/j_{w1}. These expressions constitute an improvement on all prior ones, most of which incorporate implicit idealizations such as $\tau/\tau_{w1}=1$, $j/j_{w1}=1$, and $a_0=(a_0)_{\text{laminar}}$.

Unconfined flow along a flat plate

The development of exact integral expressions for momentum and heat transfer in turbulent flow along a flat plate is straightforward, but their implementation to obtain numerical values is much more difficult than for confined flows. The time-averaged differential force-momentum balance itself may be written as

$$\frac{\tau}{\tau_w}\left[1-(\overline{u'v'})^{++}\right]=\frac{du^+}{dy^+} \tag{86}$$

and integrated formally from $u^+=0$ at $y^+=0$ (on the surface) to obtain

$$u^+=\int_0^{y^+}\left(\frac{\tau}{\tau_w}\right)\left[1-(\overline{u'v'})^{++}\right]dy^+. \tag{87}$$

Specializing Eq. 87 for $y\to\infty$ gives

$$\left(\frac{2}{C_f}\right)^{1/2}\equiv u_\infty^+=\int_0^\infty\left(\frac{\tau}{\tau_w}\right)\left[1-(\overline{u'v'})^{++}\right]dy^+, \tag{88}$$

which, since $\tau=0$ for $y^+\geq\delta^+$, where δ is the nominal thickness of the boundary layer, may be written more explicitly as

$$\left(\frac{2}{C_f}\right)^{1/2}=u_\infty^+=\int_0^{\delta^+}\left(\frac{\tau}{\tau_w}\right)\left[1-(\overline{u'v'})^{++}\right]dy^+. \tag{89}$$

The ill-defined quantity δ^+ appears to be avoided in Eq. 88, but is implicit in τ/τ_w, which is not known *a priori*. The second integration required to determine $u_m^+=(2/f)^{1/2}$ for confined flow in a channel is avoided for unconfined flow over a flat plate, but at the expense of the unknown δ^+ or its equivalent in the limiting behavior of τ/τ_w.

Churchill (1994) represented values of τ/τ_w, as determined from experimental measurements and direct numerical simulations of $\overline{u'v'}$ and u, with the purely empirical expression

$$\frac{\tau}{\tau_w}=\cos^2\left\{\frac{\pi y^+}{2\delta^+}\right\} \tag{90}$$

and thereby proposed a correlating equation for $(\overline{u'v'})^+$ that may be adapted for $(\overline{u'v'})^{++}$ as follows:

$$\overline{(u'v')}^{++}\cos^2\left\{\frac{\pi y^+}{2\delta^+}\right\} = \left(\left[0.7\left(\frac{y^+}{10}\right)^3\right]^{-8/7} + \left|\cos^2\left\{\frac{\pi y^+}{2\delta^+}\right\} - \frac{2.439}{y^+} - \left(\frac{1.175\pi}{\delta^+}\right)\sin\left\{\frac{\pi y^+}{\delta^+}\right\}\right|^{-8/7}\right)^{-7/8}. \quad (91)$$

The predictions of Eq. 91 become unbounded as $y^+ \to \delta^+$, owing to the functional failure of Eq. 90 in that limit. The magnitude of this discrepancy increases as δ^+ decreases. As a practical recourse, $\overline{(u'v')}^+$ and $\overline{(u'v')}^{++}$ may simply be taken to be zero in that regime.

Churchill (1993) used the following correlating equation for the velocity distribution in the turbulent core ($30 \le y^+ \le \delta^+$)

$$u^+ = 5.0 + 2.439\ln\{y^+\} + 2.35\sin^2\left\{\frac{\pi y^+}{2\delta^+}\right\} \quad (92)$$

to derive, by means of an integral balance for momentum

$$\left(\frac{2}{C_f}\right)^{1/2} = 4.216 + 2.439\ln\left\{\frac{(C_f/2)\,Re_x}{1 - 11.264(C_f/2)^{1/2} + 43.05(C_f/2)}\right\}. \quad (93)$$

Setting $y^+ = \delta^+$ in Eq. 92 gives

$$\left(\frac{2}{C_f}\right)^{1/2} = 7.35 + 2.439\ln\{\delta^+\}. \quad (94)$$

The value of $(2/C_f)^{1/2}$ may be determined for any specified value of Re_x by iterative solution of Eq. 93, and then the value of δ^+ from Eq. 94. In view of Eqs. 92 and 93, the need for Eqs. 87 and 88 might be questioned, particularly with δ^+ taken from Eqs. 93 and 94. However, Eq. 87 predicts the velocity distribution below the limit of $y^+ > 30$ of Eq. 92, and perhaps more accurately for all values, while Eq. 88 provides a more direct basis of comparison for heat transfer and for other geometries.

The κ-ϵ and κ-ϵ-$\overline{u'v'}$ models are valid alternatives for the prediction of $\overline{u'v'}$, u^+, $u_\infty^+ = (2/C_f)^{1/2}$, δ^+, and τ/τ_w for unconfined flow over a flat plate, but the added differential balances invoke considerable empiricism.

Equation 41 is directly applicable for heat transfer from a flat plate in unconfined flow, and integration gives the following exact expression for the temperature distribution:

$$T^+ = \int_0^{y^+}\left(\frac{j}{j_w}\right)\left[1 - \overline{(u'v')}^{++}\right]\left(\frac{Pr_T}{Pr}\right)dy^+. \quad (95)$$

Specialization of Eq. 95 for $y^+ \to \infty$ results in

$$T_\infty^+ \equiv \frac{Re_x(C_f/2)^{1/2}}{Nu_x} = \int_0^\infty\left(\frac{j}{j_w}\right)\left[1 - \overline{(u'v')}^{++}\right]\left(\frac{Pr_T}{Pr}\right)dy^+, \quad (96)$$

which, since $j = 0$ for $y^+ > \delta_E^+$, where δ_E is the thickness of the thermal boundary layer, may be expressed in the more explicit form

$$Nu_x = \frac{Re_x(C_f/2)^{1/2}}{\int_0^{\delta_E^+}\left(\frac{j}{j_w}\right)\left[1 - \overline{(u'v')}^{++}\right]\left(\frac{Pr_T}{Pr}\right)dy^+}. \quad (97)$$

The dimensionless thermal boundary layer thickness and heat-flux density distribution j/j_w in Eq. 91 are unknown *a priori*. For $Pr \to \infty$, $\delta_E^+ \ll \delta^+$; for $Pr \to 0$, $\delta_E^+ \gg \delta^+$; for $Pr_T = Pr$, $\delta_E^+ \cong \delta^+$. Reichardt (1951) deduced that $j/j_w \cong \tau/\tau_w$, while $j/j_w = 1$ has been postulated in the derivation of a number of approximate solutions for Nu_x. It may be noted that, in constrast to confined flow, postulating $Pr_T = Pr$, $\delta_E^+ = \delta^+$ and $j/j_w = \tau/\tau_w$ reduces Eq. 97 to the Reynolds analogy:

$$Nu_x\{1\} = Re_x(C_f/2). \quad (98)$$

In order to permit the general implementation of Eq. 97, generalized correlating equations are required for j/j_w and δ_E^+ as well as for $\overline{(u'v')}^{++}$, τ/τ_w, and δ^+. Accurate measurements or direct numerical simulations of T and $\overline{T'v'}$ or the equivalent over a wide range of values of Re_x and Pr are needed to support the construction of such expressions. An alternative and interim approach is to use a κ-ϵ-$\overline{u'v'}$-$\overline{T'v'}$ model, but this procedure is at the expense of considerable empiricism and computing.

Flow in two-dimensional channels

The turbulent-shear-stress, eddy-viscosity, mixing-length, κ-ϵ, and κ-ϵ-$\overline{u'v'}$ models all fail for two-dimensional channels because of the ever-present secondary motion.

Summary and Conclusions

1. The representation of the time-averaged differential force-momentum balance in terms of the fraction of the local shear stress due to turbulence, $\overline{(u'v')}^{++} \equiv -\rho\overline{u'v'}/\tau$, and the total shear stress ratio, τ/τ_w, leads to exact integral expressions for the velocity distribution and the friction factor or drag coefficient in all one-dimensional, fully developed, turbulent flows. These expressions are simpler and potentially more accurate for numerical evaluations than those based on other models or in terms of other variables.

2. Churchill and Chan (1995b) proposed the representation of turbulent flows in terms of $\overline{(u'v')}^+ \equiv -\rho\overline{u'v'}/\tau_w$, which they showed to be decisively superior to the eddy viscosity and the mixing length in all respects. The further improvement provided by the modified model described herein arises from its purely local character, and manifests itself most clearly in terms of algebraic simplifications.

3. One of the findings of the current investigation is that the eddy viscosity and the mixing length are completely independent of their mechanistic origin. For example, $\mu_t/\mu = \overline{(u'v')}^{++}/[1-\overline{(u'v')}^{++}]$, and thereby may be interpreted as simply a symbol for the ratio of the shear stress due to turbulence to that due to viscosity. The frequent dismissal of the eddy viscosity and the mixing-length as "less theoretical" than other time-averaged characteristics of turbulent flow is unjustified. On the other hand, they are inferior in applications in every sense to $\overline{(u'v')}^{++}$ as well as to $\overline{(u'v')}^{+}$. In particular, the mixing length is unbounded at the center line of a round tube and the central plane between parallel plates, while both the mixing length and the eddy viscosity are unbounded at one location within the fluid and negative over an adjacent range in all flows, including annuli, in which the shear stress is not equal at opposing points on the walls. Also, correlating equations for the eddy viscosity and the mixing length are more complex, as are expressions for their implementation.

4. The joint failures of the eddy-viscosity and mixing-length models for some flows carry over to the κ-ϵ model since it functions by predicting one of the other of these quantities. This failure does not carry over to the κ-ϵ-$\overline{u'v'}$ models, but they introduce considerable empiricism and are valid only for the turbulent core.

5. The new integral formulations require a theoretical expression or a correlating equation for the shear stress ratio τ/τ_w and a correlating equation for the dimensionless shear stress $\overline{(u'v')}^{++}$. Exact expressions are known for the shear-stress ratio for flow in round tubes, between parallel plates, in planar Couette flow, and in annuli, although that for the latter incorporates a quantity that is not known *a priori*. An exact expression for the shear stress distribution in unconfined flow along a flat plate is not known, but an empirical correlating equation has been devised. Correlating equations for $\overline{(u'v')}^{++}$ with a theoretical structure, but with empirical coefficients and exponents, have been developed for all of the geometries just mentioned except for an annulus, and one appears feasible for that case.

6. The time-averaged differential energy balance for fully developed convection in one-dimensional fully developed turbulent flow may be expressed exactly in terms of the fraction of the local heat-flux density due to turbulence, namely $\overline{(T'v')}^{++} = \rho c_p \overline{T'v'}/j$, and the heat flux density ratio j/j_w. For uniform heating on the wall or walls of a channel j/j_w may in turn be expressed in terms of exact integrals of $\overline{(u'v')}^{++}$, but for unconfined flow such a relationship must be determined indirectly.

7. The most surprising finding of this investigation is that

$$\frac{Pr_t}{Pr} = \left(\frac{\overline{(u'v')}^{++}}{1-\overline{(u'v')}^{++}}\right)\left(\frac{1-\overline{(T'v')}^{++}}{\overline{(T'v')}^{++}}\right)$$

and thereby that

$$\frac{Pr_T}{Pr} = \frac{1-\overline{(T'v')}^{++}}{1-\overline{(u'v')}^{++}}.$$

The turbulent Prandtl number and the total Prandtl number have generally been disparaged as empirical artifacts of the eddy diffusional model. However, it is apparent that Pr_t/Pr may be interpreted as the ratio of shear stress due to turbulence to that due to viscosity, divided by the corresponding ratio for the heat-flux density, while Pr_T/Pr may be interpreted simply as the ratio of the fraction of the heat-flux density due to thermal conduction to the fraction of the shear stress due to the viscosity. Both quantities are obviously independent of their heuristic diffusional origin.

8. It follows from the preceding interpretation of Pr_T that $(Pr_T/Pr)[1-\overline{(u'v')}^{++}]$ may be substituted for $1-\overline{(T'v')}^{++}$ in the differential energy balance, and hence that a correlation for Pr_T may be utilized rather than developing one for $\overline{(T'v')}^{++}$.

9. The experimental measurements of Abbrecht and Churchill (1960) for a round tube appear to confirm the common speculation that Pr_t (and therefore Pr_T) is independent of the thermal boundary condition and hence depends only on the field of flow and the Prandtl number.

10. The integral expressions presented herein for u^+, T^+, f, C_f, Nu_D, and Nu_x are exact and may be expected to yield results of improved accuracy because of their simplicity and the relative reliability of the expressions for the parameters therein. The principal uncertainty in the prediction of heat transfer is associated with that of the correlations for Pr_t and Pr_T.

11. Although the eddy viscosity, the eddy conductivity, and the mixing length appear to be supplanted by the new models described herein, they have served a useful role over much of the past century. Furthermore, their intuitive derivation has even been substantiated in part.

12. Finally, it is remarkable that such complex processes as turbulent flow and heat transfer can be described with such simplicity, generality, and relative accuracy as that provided by the integrals herein of $\overline{(u'v')}^{++}$.

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Notation

a = radius of tube
$a^+ = a(\tau_w \rho)^{1/2}/\mu$
A = empirical coefficient
b = half-spacing between plates
$b^+ = b(\tau_w \rho)^{1/2}/\mu$
C_f = drag coefficient = $2\tau_w/\rho u_\infty^2$
c_p = specific heat capacity at constant pressure
D = diameter of tube or hydraulic diameter of parallel-plate channel or annulus
e = roughness
f = Fanning friction factor = $2\tau_w/\rho u_m^2$
j = heat-flux density
j_w = heat-flux density at wall
j_{w1} = heat-flux density of inner wall of annulus
k = molecular conductivity
k_t = eddy conductivity
k_T = total conductivity = $k + k_t$
l = mixing length
$l^+ = l(\tau_w \rho)^{1/2}/\mu$
Nu_D = Nusselt number = $j_w D/k(T_w - T_m)$
Nu_x = Nusselt number for flat plate = $j_w x/k(T_w + T_\infty)$
r = radial coordinate
$r^+ = r(\tau_w \rho)^{1/2}/\mu$

$R = r/a$
Re_D = Reynolds number of channel = $Du_m \rho/\mu$
Re_x = Reynolds number for flat plate = $xu_\infty \rho/\mu$
T = time-averaged temperature
T' = fluctuation in temperature
T_c = temperature at central plane
T_m = mixed-mean temperature
T_w = wall temperature
T_∞ = free-stream temperature
T_1 = temperature at wall 1
T_2 = temperature at wall 2
u = time-averaged velocity in x-direction
$u^+ = u(\rho/\tau_w)^{1/2}$
u' = fluctuation in velocity
u_m = mixed-mean velocity
$u_m^+ = u_m(\rho/\tau_w)^{1/2} = (2/f)^{1/2}$
u_∞ = free stream velocity
v' = fluctuation in velocity in y-direction
x = coordinate in axial direction
y = coordinate normal to wall
$y^+ = y(\tau_w \rho)^{1/2}/\mu$
$Y = y/a$
α = arbitrary exponent in Eq. 2
δ_E = thermal boundary layer thickness
$\delta^+ = \delta(\tau_w \rho)^{1/2}/\mu$
$\gamma = (j/j_w)/(\tau/\tau_w) - 1$
μ = dynamic molecular viscosity
ρ = specific density
$\phi\{z\}$ = function of z

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